

A Two-Dimensional Maximum Likelihood Parameter Estimation of Γ - Γ Distribution for Free Space Optical Channels with Weak Turbulence Conditions

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Abstract—We present an approach for maximum likelihood (ML) parameter estimation of the Gamma-Gamma (Γ - Γ) distribution in the weak turbulence conditions of the free space optical (FSO) channels. A two-dimensional ML (2DML) estimation approach is deployed to extend our one-dimensional Γ - Γ parameter estimation (1DML) method proposed in [1]. To achieve the 2D estimation, an explicit closed form expression between the Γ - Γ parameters is extracted, where the constant factors of the expression are obtained using genetic algorithm (GA). The proposed 2DML estimation is compared with the modified method of moments based on a convex optimization (modified MOM/CVX). The numerical results demonstrate that 2DML outperforms modified MOM/CVX in terms of the mean square error of the estimation and the variance of the estimators. Moreover, the convergence rate of the 2DML method is high and not sensitive to the starting points.

Keywords-Maximum Likelihood Estimation; Free Space Optical (FSO); Gamma-Gamma Distribution; Genetic Algorithm

I. INTRODUCTION

Free space optical (FSO) communication is advantageous in the sense that it provides high data rate, uses unregulated license free spectrum, and is easy-to-implement. Moreover it has inherent security due to the narrow optical beams. However, the reliability of FSO communications is affected by the environment where the optical beam is propagated. One of the main impairments of the FSO channels is the optical turbulence, caused by the variations of the refractive index along the propagation path due to the temperature and pressure changes [2]. The optical turbulence causes irradiance fluctuations (scintillation)

of the received signals and affects the FSO communications performance significantly [3], [4].

The Gamma-Gamma (Γ - Γ) distribution is a two-parameter model which is accurate for a wide range of FSO channel turbulence conditions [5]. The parameters of the Γ - Γ distribution (α and β) are related to some atmospheric parameters, such as, the refractive index structure parameter or Rytov variance (σ_r^2) which describes the strength of the optical turbulence. Based on the values of Rytov variance, three levels of turbulence strength are identified for FSO channels: weak ($\sigma_r^2 \leq 0.3$), moderate ($0.3 < \sigma_r^2 \leq 5$),

and strong ($\sigma_1^2 > 5$) [6]. Finding a channel distribution which fits the FSO channel statistics is required in many applications, such as, the FSO communications protocols performance evaluation or design [7]. To achieve this goal, parameter estimation of the Γ - Γ distribution from observed data is investigated in the literature.

Method of moments (MOM) is a simple method to estimate the Γ - Γ distribution parameters based on the sample moments. The MOM estimator is not accurate due to the problem of outlier samples. The fractional moments method (FMOM) is another moment based estimator which employs the fractional moments to reduce the effect of outlier samples. However, the FMOM results in invalid values of estimations for $\sigma_1^2 < 1$ [6]. To address this problem, a suboptimal estimation (MOM/CVX) is proposed in [6] which obtains good estimations of β . The estimation of α needs some improvement, so a modified MOM/CVX method is proposed [6]. However, in $\sigma_1^2 < 1$, the performance of modified MOM/CVX is reduced. A one-dimensional Γ - Γ parameter estimation (1DML) method is proposed in [1]. Maximum likelihood-based estimation methods are asymptotically efficient, but estimating both Γ - Γ parameters at the same time is challenging due to the difficulties in obtaining ML estimates and solving nonlinear ML equations.

In this paper, we propose an approach to determine a two-dimensional maximum likelihood (2DML) parameter estimation of the Γ - Γ distribution which estimates α and β simultaneously for the weak turbulence conditions, i.e., $\sigma_1^2 < 1$. We extract the ML equations and approximate them to obtain Maximum likelihood (ML) estimate. Then, we use genetic algorithm (GA) to optimize the derived expression in terms of the mean square error (MSE). The numerical results demonstrate that 2DML method is more accurate in comparison to modified MOM/CVX.

The rest of the paper is organized as follows. In section II, the Γ - Γ distribution is introduced briefly. In section III, the 2DML estimator is presented. In section IV, the GA is deployed to optimize the 2DML method. Numerical results are presented in section V, and the paper is concluded in section VI.

II. GAMMA-GAMMA DISTRIBUTION

The Γ - Γ distribution is a modulation based model where the irradiance fluctuations are governed by two independent Γ distributions. The Γ - Γ distribution is a good fit to the optical channel irradiance fluctuations over all turbulence conditions with the given probability density function [5]:

$$p(I; \alpha, \beta) = \frac{2(\alpha\beta)^{\frac{1}{2}(\alpha+\beta)}}{\Gamma(\alpha)\Gamma(\beta)} I^{\frac{1}{2}(\alpha+\beta)-1} K_{\alpha-\beta}(2\sqrt{\alpha\beta}I) \quad (1)$$

where $I > 0$ denotes the irradiance (intensity) of the optical wave, $K_v(\cdot)$ is the modified Bessel function of the second kind of order v . $\alpha > 0$ and $\beta > 0$ are the shape parameters and directly related to atmospheric turbulence conditions. With the assumption of the plane wave data, the shape parameters are [8]:

$$\alpha \cong \left[\exp \left(\frac{.49\sigma_1^2}{(1+1.11\sigma_1^{12/5})^{7/6}} \right) - 1 \right]^{-1} \quad (2)$$

$$\beta \cong \left[\exp \left(\frac{.51\sigma_1^2}{(1+.69\sigma_1^{12/5})^{5/6}} \right) - 1 \right]^{-1} \quad (3)$$

where $\sigma_1^2 = 1.23C_n^2 k^{7/6} L^{11/6}$ is Rytov variance, C_n^2 is the refractive index structure parameter which is constant for a horizontal path communication link, $k = 2\pi/\lambda$ is the wave number and L is the link range.

III. 2DML PARAMETER ESTIMATION OF Γ - Γ DISTRIBUTION

In this section, we propose an approach to determine ML parameter estimation of the Γ - Γ distribution to extend our previous contribution [1] to 2DML parameter estimation. To achieve the ML equations, we rewrite the Γ - Γ distribution as follows:

$$p(I) = \frac{\alpha^\alpha \beta^\beta \Gamma^{\beta-1}}{\Gamma(\alpha)\Gamma(\beta)} \int_0^\infty x^{\alpha-\beta-1} \exp\left(-\alpha x - \frac{\beta I}{x}\right) dx \quad (4)$$

A likelihood function for α and β can be presented directly based on N independent and identically distributed (*iid*) observed data, $I = \{I_1, I_2, \dots, I_N\}$, and the probability density function given in (4):

$$L(\alpha, \beta) = \frac{\alpha^{N\alpha} \beta^{N\beta}}{\Gamma^N(\alpha)\Gamma^N(\beta)} \prod_{i=1}^N (I_i^{\beta-1}) \prod_{i=1}^N \left(\int_0^\infty x^{\alpha-\beta-1} \exp\left(-\alpha x - \frac{\beta I_i}{x}\right) dx \right) \quad (5)$$

Accordingly the log-likelihood function is

$$\begin{aligned} \log L(\alpha, \beta) &= N[\alpha \log(\alpha) + \beta \log(\beta) - \log(\Gamma(\alpha))] \\ &\quad - N \log(\Gamma(\beta)) + (\beta - 1) \sum_{i=1}^N \log(I_i) \\ &\quad + \sum_{i=1}^N \log \left\{ \int_0^\infty x^{\alpha-\beta-1} \exp\left(-\alpha x - \frac{\beta I_i}{x}\right) dx \right\} \quad (6) \end{aligned}$$

Taking partial derivative of the log-likelihood function with respect to α yields:

$$\begin{aligned} \frac{\partial}{\partial \alpha} (\log L(\alpha, \beta)) &= N(\log(\alpha) + 1 - \psi(\alpha)) \\ &\quad + \sum_{i=1}^N \frac{\frac{\partial}{\partial \alpha} \left\{ \int_0^\infty x^{\alpha-\beta-1} \exp\left(-\alpha x - \frac{\beta I_i}{x}\right) dx \right\}}{\int_0^\infty x^{\alpha-\beta-1} \exp\left(-\alpha x - \frac{\beta I_i}{x}\right) dx} \quad (7) \end{aligned}$$

Now, we compute the derivative of the integral term in (7):

$$\frac{\partial}{\partial \alpha} \left\{ \int_0^\infty x^{\alpha-\beta-1} \exp\left(-\alpha x - \frac{\beta I_i}{x}\right) dx \right\}$$



$$= \int_0^\infty \log(x)x^{\alpha-\beta-1} \exp\left(-\alpha x - \frac{\beta I_i}{x}\right) dx - \int_0^\infty x^{\alpha-\beta} \exp\left(-\alpha x - \frac{\beta I_i}{x}\right) dx. \quad (8)$$

The first integral of the right hand side of (8) is calculated by the following theorem [9].

Theorem 1: Let T be an open subset of \mathbb{R} and X be a measure space. Suppose $h: T \times X \rightarrow \mathbb{R}$ satisfies the following conditions:

- (1) $h(t, x)$ is a measurable function of t for each $t \in T$.
- (2) For almost all $x \in X$, the derivative $\partial h(t, x)/\partial t$ exists for all $t \in T$.
- (3) There is an integrable function $\theta: X \rightarrow \mathbb{R}$ such that $|\partial h(t, x)/\partial t| \leq \theta(x)$ for $t \in T$.

Then,

$$\frac{d}{dt} \int_T h(t, x) dx = \int_T \frac{\partial}{\partial t} h(t, x) dx. \quad (9)$$

Assume

$$h(t, x) = x^{\alpha-\beta-1+t} \exp\left(-\alpha x - \frac{\beta I_i}{x}\right). \quad (10)$$

The first integral of the right hand side of (8) is

$$\begin{aligned} & \int_0^\infty \log(x)x^{\alpha-\beta-1} \exp\left(-\alpha x - \frac{\beta I_i}{x}\right) dx \\ &= \left(\int_0^\infty \frac{\partial}{\partial t} \left(x^{\alpha-\beta-1+t} \exp\left(-\alpha x - \frac{\beta I_i}{x}\right) \right) dx \right)_{t=0} \\ &= \frac{d}{dt} \left(\int_0^\infty x^{\alpha-\beta-1+t} \exp\left(-\alpha x - \frac{\beta I_i}{x}\right) dx \right)_{t=0} \\ &= \frac{d}{dt} \left(2\sqrt{\frac{\beta I_i}{\alpha}} K_{\alpha-\beta+t} \left(2\sqrt{\alpha\beta I_i} \right) \right)_{t=0} \\ &= 2\sqrt{\frac{\beta I_i}{\alpha}}^{\alpha-\beta} \left\{ \log \sqrt{\frac{\beta I_i}{\alpha}} K_{\alpha-\beta} \left(2\sqrt{\alpha\beta I_i} \right) + \frac{\partial}{\partial t} \left(K_t \left(2\sqrt{\alpha\beta I_i} \right) \right)_{t=\alpha-\beta} \right\}. \quad (11) \end{aligned}$$

The second integral of the right hand side of (8) is:

$$\begin{aligned} & \int_0^\infty x^{\alpha-\beta} \exp\left(-\alpha x - \frac{\beta I_i}{x}\right) dx \\ &= 2\sqrt{\frac{\beta I_i}{\alpha}}^{\alpha-\beta+1} K_{\alpha-\beta+1} \left(2\sqrt{\alpha\beta I_i} \right). \quad (12) \end{aligned}$$

Using (7), (8), (11) and (12), we have

$$\frac{\partial}{\partial \alpha} (\log L(\alpha, \beta)) = N(\log(\alpha) + 1 - \psi(\alpha))$$

$$+ \sum_{i=1}^N \left\{ \log \sqrt{\frac{\beta I_i}{\alpha}} + \frac{\frac{\partial}{\partial p} \left(K_p \left(2\sqrt{\alpha\beta I_i} \right) \right) \Big|_{p=\alpha-\beta}}{K_{\alpha-\beta} \left(2\sqrt{\alpha\beta I_i} \right)} \right\} - S_1 = 0, \quad (13)$$

where $\psi(\cdot)$ is digamma function and

$$S_1 = \sum_{i=1}^N \sqrt{\frac{\beta I_i}{\alpha}} \frac{K_{\alpha-\beta+1} \left(2\sqrt{\alpha\beta I_i} \right)}{K_{\alpha-\beta} \left(2\sqrt{\alpha\beta I_i} \right)}. \quad (14)$$

The partial derivative of the log-likelihood function with respect to β is calculated with the same approach as the one's of α :

$$\begin{aligned} & \frac{\partial}{\partial \beta} (\log L(\alpha, \beta)) = N(\log(\beta) + 1 - \psi(\beta)) + \sum_{i=1}^N \log(I_i) \\ & - \sum_{i=1}^N \left\{ \log \sqrt{\frac{\beta I_i}{\alpha}} + \frac{\frac{\partial}{\partial p} \left(K_p \left(2\sqrt{\alpha\beta I_i} \right) \right) \Big|_{p=\alpha-\beta}}{K_{\alpha-\beta} \left(2\sqrt{\alpha\beta I_i} \right)} \right\} - S_2 = 0, \quad (15) \end{aligned}$$

where

$$S_2 = \sum_{i=1}^N \sqrt{\frac{\alpha I_i}{\beta}} \frac{K_{\alpha-\beta-1} \left(2\sqrt{\alpha\beta I_i} \right)}{K_{\alpha-\beta} \left(2\sqrt{\alpha\beta I_i} \right)}. \quad (16)$$

Since there are no closed form solutions to the ML equations (13) and (15), 2D numerical search methods, such as, Nelder-Mead (NM) algorithm can be used to obtain the maximum likelihood estimators. The Γ - Γ Log-likelihood function is flat for the large values of the shape parameters as shown in Fig. 1. Therefore, the NM algorithm has difficulty to find the true maximum, and cannot estimate well the shape parameters for $\sigma_i^2 < 1$. Moreover the 2D evaluation techniques are computationally expensive. Thus, we present an alternative approach to obtain the 2DML parameter estimation.

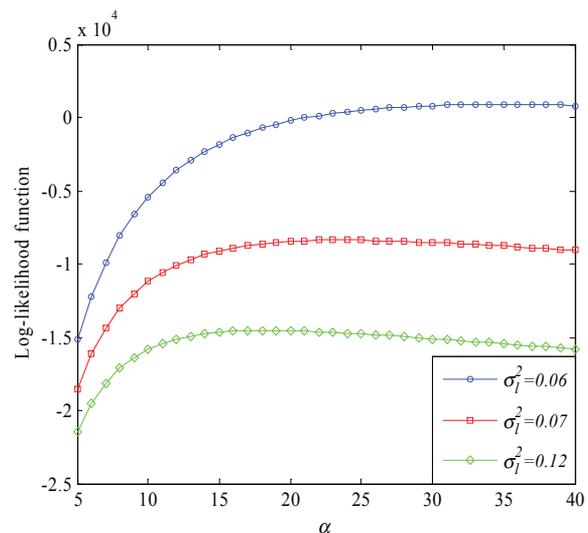


Fig. 1. Log-likelihood functions versus α for different values of Rytov variance



In [1], α is estimated based on the 1DML parameter estimation. The method uses generalized Newton method and the expectation maximization (EM) algorithm to estimate α :

$$\frac{1}{\alpha_{k+1}} = \frac{1}{\alpha_k} + \frac{-\psi(\alpha_k) + \log(\alpha_k) + 1}{\alpha_k - \alpha_k^2 \psi'(\alpha_k)} + \frac{\frac{1}{N} \sum_{i=1}^N E_{x|I; \alpha_k} [\log(x_i) - x_i]}{\alpha_k - \alpha_k^2 \psi'(\alpha_k)} \quad (17)$$

where α_{k+1} is the new estimate, α_k is the current estimate, $\psi(\cdot)$ is trigamma function, and

$$E_{x|I; \alpha_k} [\log(x_i) - x_i] = -\frac{\sqrt{\beta I_i} K_{\alpha_k - \beta - 1}(2\sqrt{\alpha_k \beta I_i})}{\sqrt{\alpha_k} K_{\alpha_k - \beta}(2\sqrt{\alpha_k \beta I_i})} + \log \sqrt{\frac{\beta I_i}{\alpha_k}} + \frac{\frac{\partial}{\partial p} K_p(2\sqrt{\alpha_k \beta I_i})|_{p=\alpha_k - \beta}}{K_{\alpha_k - \beta}(2\sqrt{\alpha_k \beta I_i})} \quad (18)$$

Now, we need to derive a relationship between the shape parameters to provide a 2D estimation. From (13) and (15), we obtain:

$$N [\log(\alpha) + \log(\beta) + 2 - \psi(\alpha) - \psi(\beta)] + \sum_{i=1}^N \log(I_i) = S_1 + S_2 \quad (19)$$

The terms S_1 and S_2 are very close to each other for $\sigma_I^2 < 1$. Table 1 shows the ratio of these terms for different values of the shape parameters corresponding to the σ_I^2 for three sample sizes, N . Therefore, (19) is restated as follows:

$$N [\log(\alpha) + \log(\beta) + 2 - \psi(\alpha) - \psi(\beta)] + \sum_{i=1}^N \log(I_i) = 2S_1 \quad (20)$$

To achieve a relationship between the shape parameters, S_1 should be converted to the polynomial expression using the following approximations [10]:

$$K_\nu(z) \approx \sqrt{\frac{\pi}{2z}} e^{-z} \left(1 + \frac{4\nu^2 - 1}{8z} \right), \quad (21)$$

$$\psi(z) \approx \log(z) - \frac{1}{2z} \quad (22)$$

The approximations are close to their true values for large arguments, $|z| \gg 1$. This situation is equivalent to $\sigma_I^2 < 1$. Using the above approximations, (14) and (20), we have:

$$2N + \sum_{i=1}^N \log(I_i) + N \left(\frac{1}{2\alpha} + \frac{1}{2\beta} \right) = 2 \sum_{i=1}^N \sqrt{\frac{\beta I_i}{\alpha}} \frac{4(\alpha - \beta + 1)^2 - 1 + 16\sqrt{\alpha \beta I_i}}{4(\alpha - \beta)^2 - 1 + 16\sqrt{\alpha \beta I_i}} \quad (23)$$

The difference of the shape parameters and the square root of multiplication of the shape parameters appear in the right hand side of (23). Moreover, due to

slight deference between the shape parameters, β can be replaced by α in the left hand side of (23). Thus, in order to reduce the degree of the derived equation and extract a closed form expression, (23) is approximated as follows:

$$2N + \sum_{i=1}^N \log(I_i) + \frac{N}{\alpha} = 2 \sqrt{\frac{\beta}{\alpha}} \frac{k(\alpha - \beta) + p\sqrt{\alpha \beta}}{l(\alpha - \beta) + p\sqrt{\alpha \beta}} \sum_{i=1}^N \sqrt{I_i}, \quad (24)$$

where k , l , and p are constant factors substituted to minimize the approximation error in the weak turbulence conditions. We apply GA to find the best values of constant factors in the next section.

To solve (24), we define the set of variables x_1 to x_6 :

$$\begin{aligned} x_1 &= 2N + \sum_{i=1}^N \log(I_i) + \frac{N}{\alpha}, \\ x_2 &= 2 \sum_{i=1}^N \sqrt{\frac{I_i}{\alpha}}, \\ x_3 &= x_2 k, \\ x_4 &= -x_1 l - x_2 p \sqrt{\alpha}, \\ x_5 &= -x_2 k \alpha + x_1 p \sqrt{\alpha}, \\ x_6 &= x_1 l \alpha. \end{aligned} \quad (25)$$

Therefore, (24) is restated by the following cubic equation in one variable:

$$x_3^2 \beta^2 + x_4 \beta + x_5 \beta^2 + x_6 = 0 \quad (26)$$

The solution of (20) results in a representation of parameter β with respect to the parameter α :

$$\beta = \left[\frac{B}{\left[(C^2 - B^3)^{1/2} - C \right]^{1/3}} - A + \left[(C^2 - B^3)^{1/2} - C \right]^{1/3} \right]^2 \quad (27)$$

where

$$\begin{aligned} A &= \frac{x_4}{3x_3}, \\ B &= \frac{x_4^2}{9x_3^2} - \frac{x_5}{3x_3}, \\ C &= \frac{x_4^3}{27x_3^3} + \frac{x_6}{2x_3} - \frac{x_4 x_5}{6x_3^2}. \end{aligned} \quad (28)$$

A 2DML parameters estimation of the Γ - Γ distribution is achieved using (17), (18) and (27) for $\sigma_I^2 < 1$.

IV. OPTIMIZATION ALGORITHM

In this section, we introduce the genetic algorithm briefly. Then, the optimization problem is presented to achieve the best values of k , l , and p in (24), which minimize the approximation error.



TABLE I. RATIO OF S₁ AND S₂

σ_i^2		.06	.07	.09	.12	.15	.19	.23	.4	.5	.7	1
α		35.03	30.21	23.82	18.25	14.936	12.16	10.38	8.43	6.79	5.02	4.39
β		32.88	28.17	21.98	16.58	13.35	10.63	8.86	6.92	5.34	3.34	2.56
S ₁	N=10000	1.0001	1.0002	1.0002	1.0005	1.0000	1.0003	0.9998	0.9998	1.0006	1.0011	1.0046
	N=25000	1.0000	1.0000	1.0001	1.0002	1.0001	0.9998	1.0001	1.0002	0.9999	0.9985	1.0009
S ₂	N=50000	1.0000	1.0001	1.0000	1.0000	0.9999	0.9999	1.0001	1.0002	1.0000	0.9991	1.0015

A. Genetic Algorithm Methodology

GA is known to be an intelligent search and optimization technique which incorporates the principles of evolution and natural selection. GA deploys a randomization search technique that avoids searching process being stopped when a local optimum is attained and continues searching the feasible region for a better local optimum [11].

In GA, each feasible solution of a problem is named chromosomes. The initial population (the first set of chromosomes) is generated randomly. Then, the initial population is improved toward the optimal solution using basic operators which generate a new population from the current chromosomes. The basic operations of GA are selection, crossover, and mutation. The selection operation chooses better chromosomes of the current generation (population) to form a population of the parent chromosomes. By crossover, some features of two selected chromosomes from the parent population are intermixed to generate new chromosomes (children). Mutation is used for changing probabilistically an arbitrary element of a chromosome to a new value hoping to find new chromosomes which may have a better fitness value.

B. Optimization Problem

The preciseness of the 2DML estimation depends on the accuracy of the derived relationship, (23), which is affected by the approximation error of equation (24). We use GA to achieve parameters k , l , and p , which minimize the approximation error.

The objective function of the optimization problem is to minimize MSE of β which is a function of constant factors, k , l , and p . The feasible space of the optimization problem contains a set of M pairs of α and β , $\{(\alpha_1, \beta_1), \dots, (\alpha_m, \beta_m), \dots, (\alpha_M, \beta_M)\}$, given in Table 1, and J is the number of iterations. The optimization problem is represented as:

$$\min_{k,l,p} \frac{1}{J^M} \prod_{m=1}^M \left[\sum_{j=1}^J (\hat{\beta}_{mj} - \beta_m)^2 \right], \quad (29)$$

where $\{\hat{\beta}_{m1} \dots \hat{\beta}_{mj} \dots \hat{\beta}_{mJ}\}$ is obtained from (27) for the m th values of the shape parameters for $j=1, \dots, J$ iteration. Note that the optimization process is different from the estimation process. Thus, we assume that the shape parameters are known to achieve the best values of the constant factors.

V. SIMULATION AND NUMERICAL RESULTS

In this section, we apply GA to determine the best values of k , l and p . Then, we implement the 2DML parameter estimation of the Γ - Γ distribution and compare its performance with the one's of the modified MOM/CVX method in terms of the MSE, variance and convergence rate metrics.

A. Genetic Algorithm Implementation and Results

To achieve an accurate relationship between the shape parameters, constant factors are obtained by GA. We consider a population of chromosomes, each encoded by an $M \times J$ matrix, where each chromosome is a feasible solution of the constant factors. An initial population is created by allocating random numbers to constant factors. We use a roulette-wheel selection operator, which selects individuals with a probability proportional to their fitness values. The crossover and the mutation operators are implemented with the uniform distribution functions.

GA is set up based on the numerical parameters given in Table II and the objective function in (29). The best constant factors obtained by the implemented GA are:

$$k = 2.23, \quad l = -1.004 \quad \text{and} \quad p = 5.263. \quad (30)$$

TABLE II. SIMULATION PARAMETERS OF GENETIC ALGORITHM

Parameters	Values
M	11
J	1000
Crossover probability	0.7
Mutation probability	0.1
Initial population	100

The accuracy of the derived relationship (21) is verified when the optimum constant factors are deployed. Fig. 2. shows that for smaller values of Rytov variance (larger values of the shape parameters), the approximated β , derived with the approximation given in (21) and (22), is very close to the true value. When the values of Rytov variance become large, the approximation error increases.



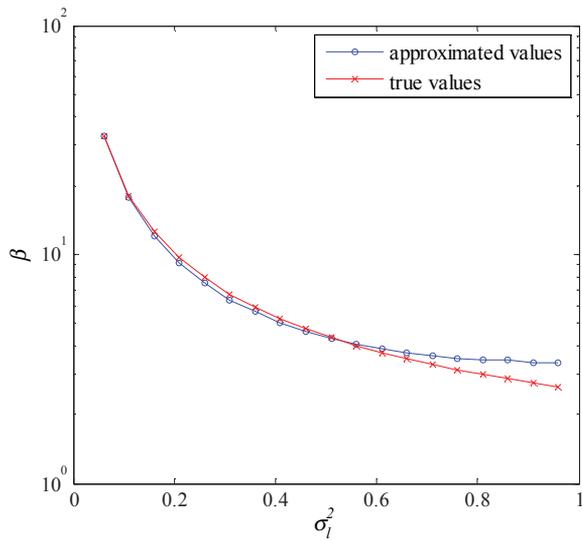


Fig. 2. Approximated β against its true values

B. 2DML Implementation and Results

We generate the Γ - Γ distributed data, for different values of the shape parameters using [1]:

$$I = \frac{\text{Gamma}_\alpha \text{Gamma}_\beta}{\alpha\beta}, \tag{31}$$

where I is the irradiance of optical wave and Gamma_α and Gamma_β are the Γ distributed random variables with shape parameters of α and β , respectively, and unit scale parameter. $\partial K_\nu(y)/\partial \nu$ is approximated by:

$$\frac{\partial}{\partial \nu} K_\nu(y) = \frac{K_{\nu+h}(y) - K_{\nu-h}(y)}{2h}, \tag{32}$$

Where $h=10^{-3}$ [1]. To evaluate the performance of the proposed method, three different sample sizes, N , over 1000 independent trials have been deployed. The maximum number of iterations and the precision of the estimation are 200 and %1, respectively, and the starting points are provided by the MOM/CVX method with $k=0.5$ for the 2DML method.

The performance of the 2DML and the modified MOM/CVX methods are compared for $\sigma_1^2 < 1$. Fig. 3 shows the MSE of 2DML and modified MOM/CVX. According to the results, the accuracy of the estimation methods improves when the sample size is increased. Moreover, two local minimums are observed in MSE curves of 2DML method corresponding to $\sigma_1^2=0.5$ and $\sigma_1^2 < 0.2$, where the approximation error is minimum (as shown in Fig. 2). Due to the high accuracy of the approximation at these points, the estimation error of α and β must be decreased compared to the other adjacent points. The MSEs of 2DML are much less than the one's of the modified MOM/CVX, specifically for $\sigma_1^2 < 0.8$. With increasing Rytov variance, the shape parameters become small and the approximations are not valid anymore. Thus, the MSE of the 2DML method increases when the Rytov variance becomes greater than 1. On the other hand, the performance of the modified MOM/CVX method significantly decreases when the Rytov variance becomes less than 0.2. Moreover, the performance of 2DML is more accurate than the one's of the modified MOM/CVX method in terms of the estimator variance as shown in Fig. 4. The difference between the MSE and variance curves is due to the approximation error. The differences are small where the approximation error is negligible, such as, in $\sigma_1^2=0.5$.

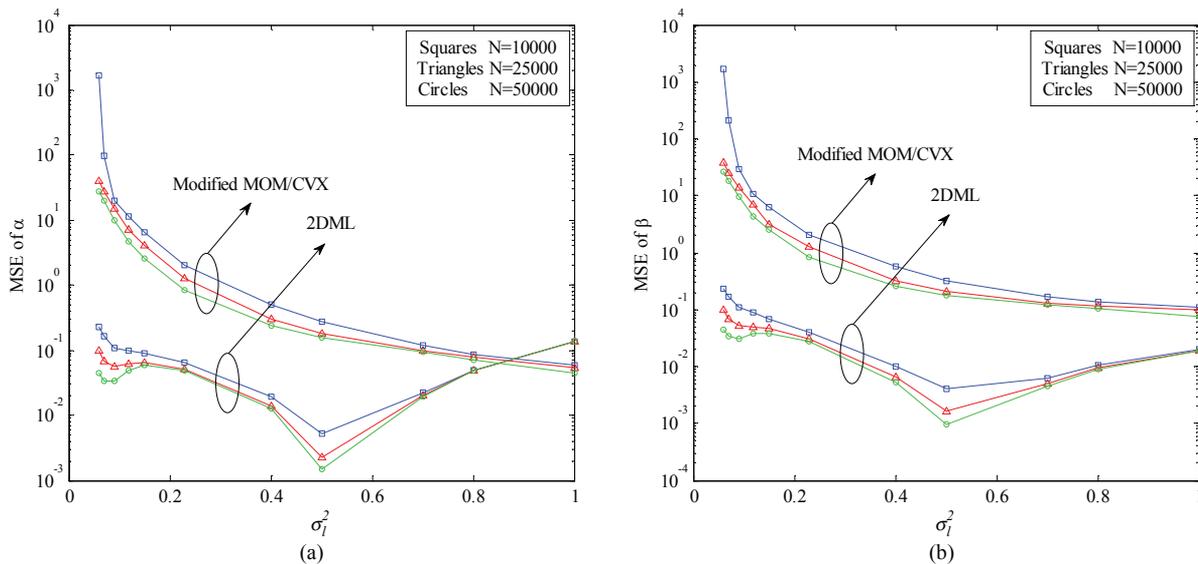


Fig. 3. (a) MSE of α and (b) MSE of β versus Rytov variance for three sample sizes



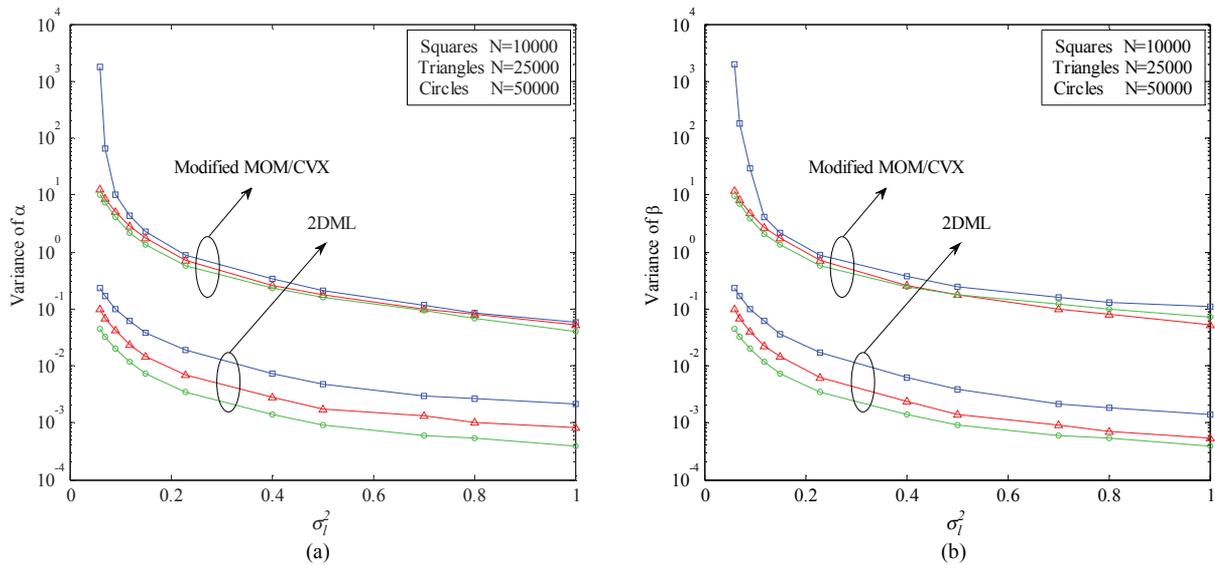
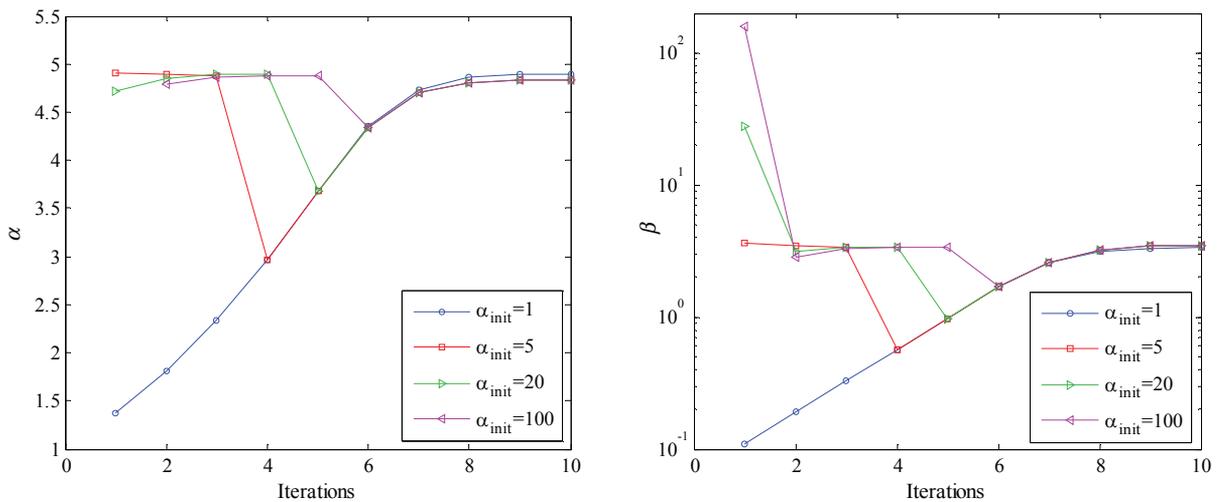


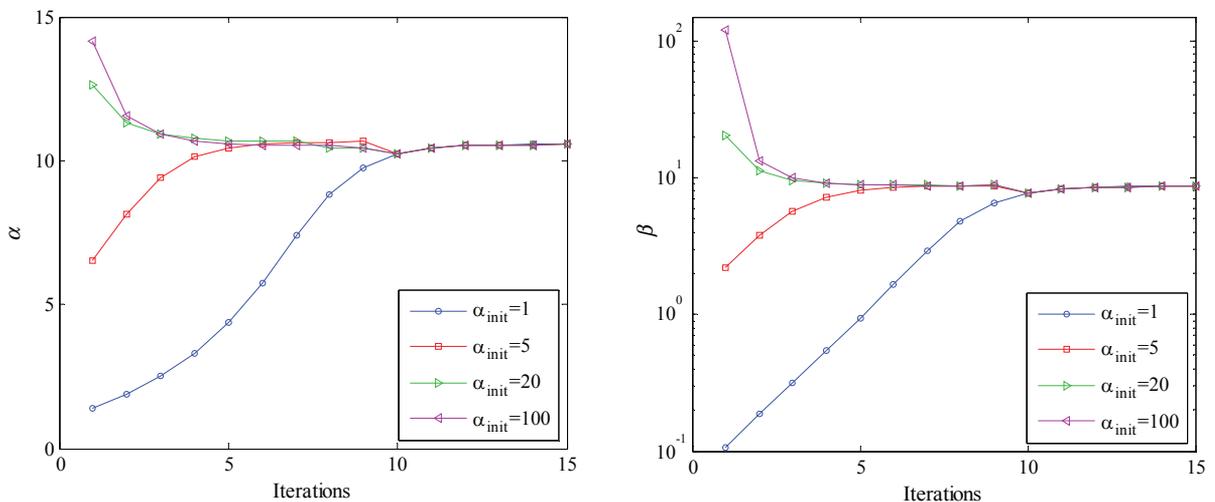
Fig. 4. (a) variance of α and (b) variance of β versus Rytov variance for three sample sizes

The convergence rate of the 2DML method for different starting points and different shape parameters are evaluated in Fig. 5. In spite of choosing random starting points, the proposed method converges quickly to the global maximum. Therefore, 2DML is a reliable and precise estimator, although

MOM/CVX cannot provide suitable starting points in the weak turbulence conditions. The reason lies behind the deployment of the generalized Newton method using a non-quadratic approximation, which decreases the dependency on the starting points [1].

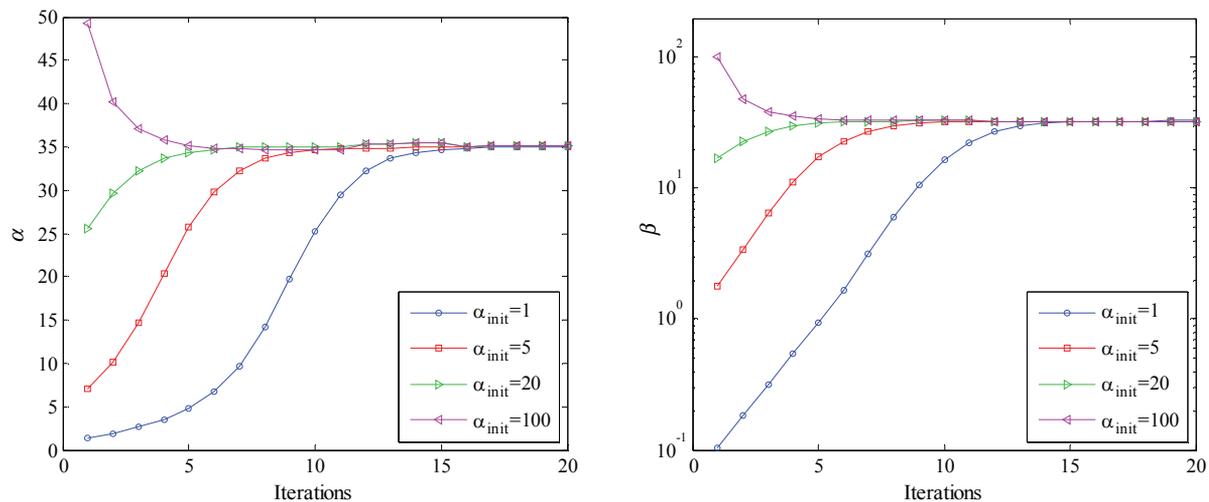


(a) $\alpha=5.02$ and $\beta=3.34$



(b) $\alpha=10.38$ and $\beta=8.86$



(c) $\alpha=35.03$ and $\beta=32.88$ Fig. 5. Convergence rate of 2DML for different initial values and three pairs of α and β parameters

VI. CONCLUSION

We proposed an approach to obtain a 2DML parameter estimation of the Γ - Γ distribution for $\sigma_l^2 < 1$. By deriving a relationship between the shape parameters, we extend the 1DML parameter estimation to 2DML. The relationship is extracted by applying some approximations and using GA.

The performance of the 2DML method is compared with the one's of the modified MOM/CVX method. The numerical results demonstrate that 2DML provides high accuracy and is less sensitive to the sample size. Moreover, the 2DML method is very slightly dependent on the starting points.

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