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User Capacity Scaling Laws for Fading Broadcast Channels Affected by Channel Estimation

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***Abstract*—It has already been shown that in rate-constrained broadcast channels, under the assumption of independent Rayleigh, Rician and Nakagami fading channels for different receivers, the user capacity (i.e. the maximum number of users that can be activated simultaneously) scales double logarithmically with the total number of users. However, to achieve the aforementioned result, it is assumed that channel state information (CSI) is perfectly known to the receivers. In practical situations, the receivers do not have access to the true CSI and they only know estimated channels. In this paper, the effects of channel estimation is analyzed on the user capacity of rate-constrained broadcast channels. In particular, the Minimum Mean Square Error (MMSE) channel estimation scheme is considered and the effects of this estimation method on the user capacity is investigated. Under the assumption of commonly used fading channels for different receivers, it is shown that the user capacity still scales double logarithmically; however, there is a gap depending on channel estimators accuracy between the upper and lower bound of the user capacity. The bound will be asymptotically tight if the variance of the channel estimator remains constant.**

***Index Terms*—User capacity,channel estimation, scaling laws, broadcast channels, fading channels, minimum-rate constraint, power allocation.**

I. Introduction

In a dynamic environment, the channel states are time-varying. In the theoretical analysis of wireless communication systems, it is usually assumed that receivers perfectly know channel state information (CSI); however, in reality, only estimated channels are available to receivers and transmitters [1]. Depending on the channel estimator used in a communication system, statistical properties of the estimated channels will change compared to the true channels. In many applications, it is of interest to analyze the effect of channel estimation on performance of a wireless communication system.

In broadcast channels, the transmitter allocates its total transmit power to different receivers according to their channel states. There is thus a fundamental trade-off between the total throughput and the minimum rate achievable for all the receivers. The basic idea is to adapt power allocation to the variations of the channel states. The transmission rate for a receiver is increased when its channel state becomes better; therefore, higher rates can be achieved using less power. This raises the issue of the trade-off between ergodic capacity and outage capacity, for which, extensive studies have been given in [2]–[4] in the context of broadcast channels.

In this paper, a rate-constrained broadcast channel is considered and an opportunistic power allocation scheme with a minimum rate constraint $R_{\min} > 0$ is utilized. For a fixed $R_{\min}$, in a time-varying fading environment, it may not be always possible for all receivers to achieve this minimum rate simultaneously. Hence, a power allocation scheme has been proposed in [5], [6] to maximize the number of active receivers supporting the minimum rate, while allocating no power to the other inactive receivers. As the number of supportable active receivers depends on the specific channel states, the asymptotic behavior should be analyzed when the total number of receivers, n, is large.

In [5], [6], under the assumption of independent Rayleigh, Rician and Nakagami fading channels for different receivers with unit noise variance, it is shown that the maximum number of active receivers scales double logarithmically with the total number of users in the system and the achieving bounds are asymptotically tight.

In this paper, the estimated channels are substituted for the true CSI and it is assumed that the variance of the channel estimator may have some variations for different users; therefore, the assumption of having independent and identically distributed (i.i.d.) channel estimators is relaxed. That is, the channel estimator variance has an upper (i.e. $\hat{\sigma}^2_{h,\max}$) and a lower bound (i.e. $\hat{\sigma}^2_{h,\min}$). It is clear that if the channel estimator variance remains constant for different users, the user capacity bound will be tight asymptotically. In this paper, the Minimum Mean Square Error (MMSE) channel estimator is considered; however, due to linearity of the observation model, other channel estimators also result in the double logarithmic user capacity scaling law.

II. Channel Model

Consider a broadcast channel with one transmitter and, n, receivers with the following channel model in the time block $t = 1, 2, \ldots, T$:

$$Y_i(t) = h_i X(t) + Z_i(t), \quad i = 1, 2, \ldots, n, \tag{1}$$

where $X(t) \in \mathbb{C}$ is the signal sent by the transmitter, and $Y_i(t) \in \mathbb{C}$ is the signal received by receiver i. The noise $Z_i(t) \in \mathbb{C}$, $i = 1, \ldots, n$, $t = 1, \ldots, T$ are assumed to be i.i.d. complex Gaussian distributed according to $\mathcal{CN}(0, \sigma^2)$. The channel gains $h_i \in \mathbb{C}$, $i = 1, \ldots, n$ are assumed to be constant during this time block.

Equivalently, the model (1) can be written as

$$Y_i'(t) = X(t) + Z_i(t)/h_i, \quad i = 1, 2, \ldots, n \tag{2}$$

where the noise $Z_i(t)/h_i$ is still complex Gaussian distributed, but with variance $\sigma^2/|h_i|^2$.

III. MMSE Channel Estimation

Coherent demodulation requires the complex channel tap $h_i = |h_i| e^{j\theta_i}$; $i = 1, \ldots, n$, to be available via perfect channel estimation. In practice, h_i is estimated from pilot symbols (i.e. s_p) extracted from the pilot tone transmitting simultaneously with the signal. In this case, the channel taps can be considered constant during a symbol, that is, $h_i(t) = h_i, \quad t \leq T_s$. Note that the channel gain, h_i, is assumed to be constant during each time block but it is a random variable over different time slots.

It is well-known that the MMSE estimated channel is given by [7, Sec. 10.3]

$$\hat{h}_i = a_i^* Y_i; i = 1, \ldots, n \tag{3}$$

$$a_i = \frac{\mathbf{E}\left\{|h_i|^2\right\} s_p}{\mathbf{E}\left\{|h_i|^2\right\} |s_p|^2 + \sigma^2} \tag{4}$$

or equivalently, with respect to the Signal-to-Noise Ratio (SNR), we have

$$a_i = \frac{SNR_p}{SNR_p + 1}\left(\frac{s_p}{|s_p|^2}\right) \tag{5}$$

where σ^2 is the noise variance and $SNR_p = |s_p|^2/\sigma^2$ is the SNR of the pilot symbol transmitted. The estimated channel becomes more accurate as the SNR increases. Hence, it can be seen that the distribution of $\hat{h_i}$; $i = 1, \ldots, n$ for Rayleigh fading and Rician fading are complex Gaussian with zero mean and mean $\hat{\mu}_i = a_i^* s_p \mu$ respectively and so both fading have same variance with $\hat{\sigma}^2_{h,i} = |a_i|^2(\sigma_h^2 |s_p|^2 + \sigma^2)$, where σ_h^2 refers to actual channel gains variance, that is

$$\hat{h}_i \sim \mathcal{CN}\left(\hat{\mu}, \hat{\sigma}^2_{h,i}\right); i = 1, \ldots, n \tag{6}$$

Note that because of linearity of the observation model, other classical estimators such as Maximum Likelihood (ML) estimators result in complex Gaussian random variable.

IV. Opportunistic Power Allocation Scheme

Let $N_i = \sigma^2/|\hat{h}_i|^2$. Without loss of generality, assume that $N_1 \leq N_2 \leq \cdots \leq N_n$. It is well known [8, Sec.14.6] that the broadcast channel (2) is stochastically degraded, and the capacity region is given by [5, Eq. (3)]. Different rates can be achieved by different power allocations. As shown by Lemma 2.1 in [5], in order to maximize the total throughput, all power should be allocated to receiver 1, which has the maximum channel gain $|\hat{h}_1|$, or the minimum equivalent noise variance N_1. Hence, the following power allocation scheme is proposed in [5], for any large integer, m, as the number of active receivers:

$$\begin{cases} \max\{m\} & (7) \\ \ln\left(1+\frac{P_1}{N_1}\right) \geq R_{\min} & (8) \\ \ln\left(1+\frac{P_i}{\sum_{j=1}^{i-1} P_j + N_i}\right) = R_{\min}; 2 \leq i \leq m & (9) \\ \sum_{i=1}^{m} P_i = P & (10) \end{cases}$$

where $R_{\min} > 0$ (in nats) is a minimum rate constraint for all active receivers. In [5], a simple recursive algorithm is also proposed to solve the aforementioned optimization problem (7)-(10). Obviously, with fixed P and $R_{\min}$, the maximum number of active receivers completely depends on the equivalent noise variance $N_i = \sigma^2/|\hat{h}_i|^2$. Let, M_n, denote the maximum number of simultaneous active receivers (out of, n, receivers) that can be supported with a rate greater than or equal to $R_{\min}$. Note that, M_n, is a random number depending on the channel gains. When the estimated channel gains $\hat{h}_i$ obey some statistical distributions, asymptotic behavior of the maximum value of m can be determined when the total number of receivers, n, becomes large. It is of interest to analyze how the user capacity of broadcast channels obtained asymptotically by Theorem 2.1 in [5] and Theorem 4.1 in [6] changes as the estimated channel gains are substituted for the true gains. The following Theorem shows the effect of MMSE channel estimation on the user capacity scaling law.

Theorem 4.1: Consider the broadcast channel in which the estimated channel gains for different receivers are distributed by (6) . For any arbitrary $\epsilon > 0$, the maximum number of active receivers, M_n, determined by (7)-(10) is bounded as

$$\mathbb{P}(\lfloor \nu_L(n) - \epsilon \rfloor \leq M_n \leq \nu_U(n) + \epsilon) \to 1, \quad \text{as } n \to \infty, \tag{11}$$

where, $\lfloor x \rfloor$ denotes the maximum integer no greater than x, n is the total number of receivers,

$$\nu_L(n) \triangleq \ln\left(\frac{\hat{\sigma}^2_{h,\min}}{\sigma^2} P \ln n\right)/R_{\min} \tag{12}$$

and

$$\nu_U(n) \triangleq \ln\left(\frac{\hat{\sigma}^2_{h,\max}}{\sigma^2} P \ln n\right)/R_{\min}. \tag{13}$$

Proof:

A. Rayleigh Fading

Consider the broadcast channel (1), with the independent gains $\hat{h}_i \sim \mathcal{CN}\left(0, \hat{\sigma}^2_{h,i}\right)$ for $i = 1, \ldots, n$. Then, $|\hat{h}_i| \sim Rayleigh\left(\sqrt{\hat{\sigma}^2_{h,i}/2}\right)$ and $|\hat{h}_i|^2 \sim \Gamma\left(1, \hat{\sigma}^2_{h,i}\right)$ for $i = 1, \ldots, n$. The Gamma cumulative distribution function is given by

$$\mathbf{F}(x; k, \theta) = \frac{\gamma(k, x/\theta)}{\Gamma(k)}$$

where $\Gamma(\cdot), \Gamma(\cdot,\cdot)$ and $\gamma(\cdot,\cdot)$ are the gamma, the upper incomplete gamma and the lower incomplete gamma functions respectively (see [6, Eqs. (13)-(15) and (23)]). Hence, for any $y > 0$,

$$\begin{aligned} \mathbf{F}(y) &= \mathbb{P}(N_i < y) = \mathbb{P}(\sigma^2/|\hat{h}_i|^2 < y) \\ &= \mathbb{P}\left(|\hat{h}_i|^2 > \sigma^2/y\right) = \exp\left(-\frac{\sigma^2}{\hat{\sigma}^2_{h,i} y}\right) \end{aligned} \tag{14}$$

It can be seen that MMSE channel estimation slightly changes the distribution function of $|h_i|^2$; $i = 1, \ldots, n$, from Exp(1) in [5] to $\Gamma\left(1, \hat{\sigma}^2_{h,i}\right)$ in this paper. The rest of the proof is basically similar to the proof of Theorem 2.1 in [5] with some modifications due to the Gamma distribution function.

For any fixed $N_0 > 0$, we can characterize the number of "good" channels with the equivalent noise variance N_i less than N_0 as the following. Let $p_0 = \mathbf{F}(N_0) = \exp(-\frac{\sigma^2}{\hat{\sigma}^2_{h,i} N_0})$. Then, with probability p_0, a channel is good. Consider a Bernoulli sequence:

$$x_i = \begin{cases} 1, & \text{with probability } p_0 \\ 0, & \text{with probability } 1 - p_0 \end{cases}$$

for $i = 1, 2, \ldots, n$. Then, the number of good channels has the same distribution as $M_n = \sum_{i=1}^{n} x_i$, which satisfies the binomial distribution $B(n, p_0)$.

For any integer $m \geq 1$, obviously,

$$\mathbb{P}(M_n \leq m-1) = \sum_{j=0}^{m-1} \binom{n}{j} p_0^j (1-p_0)^{n-j}$$

which, however, is not easy to analyze. If $m - 1 \leq np_0$, we can use the Chernoff inequality [9, page 70]:

$$\mathbb{P}(M_n \leq m-1) \leq \exp\left(-\frac{1}{2p_0}\frac{(np_0 - m + 1)^2}{n}\right).$$

Hence,

$$\mathbb{P}(M_n \geq m) \geq 1 - \exp\left(-\frac{1}{2p_0}\frac{(np_0 - m + 1)^2}{n}\right). \tag{15}$$

Now, consider the power allocations for the, m, best receivers presented in [5] given by

$$P_i = \frac{c}{\alpha^{m-i}}, \text{ for } i = 1, \ldots, m,$$

where $\alpha = e^{R_{\min}} > 1$, and $c = (1 - 1/\alpha)P$.

Next, we show that for any $\epsilon > 0$, if $m \leq \nu_L(n) - \epsilon$, $\max_{1 \leq i \leq m} N_i \leq P/\alpha^m$ holds with probability approaching one as, n, tends to infinity. Let $N_0 =$

P/α^m. Then,

$$\begin{aligned} p_0 = \mathbf{F}(N_0) &= \exp\left(-\frac{\sigma^2\alpha^m}{\hat{\sigma}^2_{h,i}P}\right) \\ &\geq \exp\left(-\frac{\sigma^2\alpha^{\nu_L(n)-\epsilon}}{\hat{\sigma}^2_{h,i}P}\right) \\ &= \exp\left(-\frac{\hat{\sigma}^2_{h,\min}}{\hat{\sigma}^2_{h,i}}\alpha^{-\epsilon}\ln n\right) = n^{-\frac{\hat{\sigma}^2_{h,\min}}{\hat{\sigma}^2_{h,i}}\lambda}, \end{aligned}$$

where $\lambda = \alpha^{-\epsilon} < 1$. Then it is obvious that as $n \to \infty$,

$$\frac{1}{2p_0}\frac{(np_0 - m + 1)^2}{n} \sim \frac{n^2p_0^2}{2np_0} = \frac{np_0}{2} \geq \frac{n^{1-\lambda}}{2} \to \infty. \quad (16)$$

Hence, by (15), the probability of $\max_{1\leq i\leq m} N_i \leq \alpha^m P$ approaches 1 as $n \to \infty$.

Therefore, we proved that as $n \to \infty$, with probability approaching 1, there are at least $m = \lfloor \nu_L(n) - \epsilon \rfloor$ good channels with $N_i \leq P/\alpha^m$, for which the minimum rate constraint is satisfied.

Next, we prove the upper bound, i.e., $m \leq \nu_U(n) + \epsilon$ holds with probability approaching 1. First in [5], we shown that for any $\delta > 0$, for sufficiently large, m, the best receiver should have the equivalent noise variance $N_1 \leq P_\delta/\alpha^m$, with $P_\delta := P + \delta$. Otherwise, if $\min_{1\leq i\leq n} N_i > P_\delta/\alpha^m$, the total power constraint or the minimum-rate constraint is violated.

Therefore, to show that

$$\mathbb{P}(M_n \leq \nu_U(n) + \epsilon) \to 1,$$

or

$$\mathbb{P}(M_n > \nu_U(n) + \epsilon) \to 0,$$

we only need to show that

$$\mathbb{P}(N_1 \leq P_\delta/\alpha^{\nu_U(n)+\epsilon}) \to 0.$$

Let $p_1 = \mathbf{F}(P_\delta/\alpha^{\nu_U(n)+\epsilon})$. Then, $(1-p_1)^n$ is the probability that all the receivers have equivalent noise variance greater than $P_\delta/\alpha^{\nu_U(n)+\epsilon}$. Hence,

$$\mathbb{P}(N_1 \leq P_\delta/\alpha^{\nu_U(n)+\epsilon}) = 1 - (1-p_1)^n, \quad (17)$$

which tends to 0 if and only if

$$\left(1 - \exp\left(-\frac{\sigma^2\alpha^{\nu_U(n)+\epsilon}}{\hat{\sigma}^2_{h,i}P_\delta}\right)\right)^n \to 1. \quad (18)$$

Since

$$\left(1 - \exp\left(-\frac{\sigma^2\alpha^{\nu_U(n)+\epsilon}}{\hat{\sigma}^2_{h,i}P_\delta}\right)\right)^{\exp\left(\frac{\sigma^2\alpha^{\nu_U(n)+\epsilon}}{\hat{\sigma}^2_{h,i}P_\delta}\right)},$$

tends to e^{-1} and (18) holds if

$$\begin{aligned} &n \cdot \exp\left(-\frac{\sigma^2\alpha^{\nu_U(n)+\epsilon}}{\hat{\sigma}^2_{h,i}P_\delta}\right) \\ =&n \cdot \exp\left(-\frac{\hat{\sigma}^2_{h,\max}}{\hat{\sigma}^2_{h,i}}\frac{P\alpha^\epsilon\ln n}{P+\delta}\right) \to 0 \end{aligned} \quad (19)$$

which holds by choosing $\delta < (\frac{\hat{\sigma}^2_{h,\max}}{\hat{\sigma}^2_{h,i}}\alpha^\epsilon - 1)P$. $\square$

B. Rician Fading

Consider the broadcast channel (1) with independent gains $\hat{h}_i \sim \mathcal{CN}\left(\hat{\mu}, \hat{\sigma}^2_{h,i}\right)$, for $i = 1, \cdots, n$; if $\tilde{h}_i \sim \mathcal{CN}\left(\hat{\mu}, 2\right)$, according to Lemma A.1, $|\hat{h}_i| = |\sqrt{\hat{\sigma}^2_h/2}\tilde{h}_i| \sim Rice(\sqrt{\hat{\sigma}^2_{h,i}/2}, \sqrt{\hat{\sigma}^2_{h,i}/2}\hat{\mu})$ and $|\tilde{h}_i|^2 \sim \mathcal{NC}\chi^2_2(\hat{\mu}^2)$ (i.e. non-central Chi-square distribution with two degrees of freedom) with the cumulative distribution function

$$\mathbf{F}_{\mathcal{NC}\chi^2_2}(x; 2, \hat{\mu}^2) = \sum_{j=0}^{\infty} e^{-\hat{\mu}^2/2}\frac{(\hat{\mu}^2/2)^j}{j!}\frac{\gamma(j+1, x/2)}{\Gamma(j+1)}$$

As $\hat{h} = \sqrt{\hat{\sigma}^2_h/2}\cdot\tilde{h}$, the distribution function of $|\hat{h}|^2$ equals to

$$\begin{aligned} &\mathbf{F}_{\mathcal{NC}\chi^2_2}(2x/\hat{\sigma}^2_h; 2, \hat{\mu}^2) \\ &= \sum_{j=0}^{\infty} e^{-\hat{\mu}^2/2}\frac{(\hat{\mu}^2/2)^j}{j!}\frac{\gamma(j+1, x/\hat{\sigma}^2_h)}{\Gamma(j+1)}. \end{aligned} \quad (20)$$

Let $N_0 = P/\alpha^m$ and $\lambda = \alpha^{-\epsilon} < 1$. Then, using [6, Eqs. (14), (15)],

$$\begin{aligned} p_0 &= \mathbf{F}(N_0) = 1 - \mathbf{F}_{\mathcal{NC}\chi^2_2}(2\frac{\sigma^2\alpha^m}{\hat{\sigma}^2_{h,i}P}; 2, \hat{\mu}^2) \\ &\geq 1 - \mathbf{F}_{\mathcal{NC}\chi^2_2}(2\frac{\sigma^2\alpha^{\nu_L(n)-\epsilon}}{\hat{\sigma}^2_{h,i}P}; 2, \hat{\mu}^2) \\ &= 1 - e^{-\hat{\mu}^2/2}\sum_{j=0}^{\infty}\frac{(\hat{\mu}^2/2)^j}{j!\,\Gamma(j+1)}\gamma\left(j+1, \frac{\hat{\sigma}^2_{h,\min}}{\hat{\sigma}^2_{h,i}}\lambda\ln n\right) \\ &= e^{-\hat{\mu}^2/2}\sum_{j=0}^{\infty}\frac{(\hat{\mu}^2/2)^j}{j!\,\Gamma(j+1)}\Gamma(j+1, \frac{\hat{\sigma}^2_{h,\min}}{\hat{\sigma}^2_{h,i}}\lambda\ln n) \\ &= n^{-\frac{\hat{\sigma}^2_{h,\min}}{\hat{\sigma}^2_{h,i}}\lambda}e^{-\hat{\mu}^2/2}\sum_{j=0}^{\infty}\frac{(\hat{\mu}^2/2)^j}{j!}\sum_{k=0}^{j}\frac{(\frac{\hat{\sigma}^2_{h,\min}}{\hat{\sigma}^2_{h,i}}\lambda\ln n)^k}{k!} \end{aligned}$$

It is clear that as $n \to \infty$,

$$\begin{aligned} &\frac{1}{2p_0}\frac{(np_0 - m + 1)^2}{n} \sim \frac{n^2p_0^2}{2np_0} = \frac{np_0}{2} \\ &\geq \frac{n^{1-\frac{\hat{\sigma}^2_{h,\min}}{\hat{\sigma}^2_{h,i}}\lambda}}{2}e^{-\hat{\mu}^2/2}\sum_{j=0}^{\infty}\frac{(\hat{\mu}^2/2)^j}{j!}\sum_{k=0}^{j}\frac{(\frac{\hat{\sigma}^2_{h,\min}}{\hat{\sigma}^2_{h,i}}\lambda\ln n)^k}{k!} \\ &\to \infty. \end{aligned} \quad (23)$$

As $m - 1 \leq np_0$, the Chernoff bound on the sum of Poisson trials can be used as (15). Hence, the probability of $\max_{1\leq i\leq m} N_i \leq P/\alpha^m$ approaches one as $n \to \infty$.

Therefore, we proved that with probability approaching one, there are at least $M_n = \lfloor \nu_L(n) - \epsilon \rfloor$ good channels for which the minimum-rate constraint is satisfied.

$$\left(1-\mathbf{F}(P_\delta/\alpha^{\nu_U(n)+\epsilon})\right)^n = \mathbf{F}_{\mathcal{NC}\chi_2^2}(2\frac{\sigma^2\alpha^{\nu_U(n)+\epsilon}}{P_\delta\hat{\sigma}^2_{h,i}};2,\hat{\mu}^2)^n \tag{21}$$

$$= \left(1-e^{-\hat{\mu}^2/2}\sum_{j=0}^{\infty}\frac{(\hat{\mu}^2/2)^j}{j!\,\Gamma(j+1)}\Gamma\left(j+1,\frac{\hat{\sigma}^2_{h,\max}P\ln n}{\hat{\sigma}^2_{h,i}\lambda P_\delta}\right)\right)^n = (1-g(n))^n \to 1.$$

$$n\,g(n) = ne^{-\hat{\mu}^2/2}\sum_{j=0}^{\infty}\frac{(\hat{\mu}^2/2)^j}{j!\,\Gamma(j+1)}\Gamma\left(j+1,\frac{\hat{\sigma}^2_{h,\max}P\ln n}{\hat{\sigma}^2_{h,i}\lambda P_\delta}\right) \tag{22}$$

$$= ne^{-\hat{\mu}^2/2}\left(\sum_{j=0}^{c\ln n}\frac{(\hat{\mu}^2/2)^j}{j!\,\Gamma(j+1)}\Gamma\left(j+1,\frac{\hat{\sigma}^2_{h,\max}P\ln n}{\hat{\sigma}^2_{h,i}\lambda P_\delta}\right)+\sum_{j=c\ln n}^{\infty}\frac{(\hat{\mu}^2/2)^j}{j!\,\Gamma(j+1)}\Gamma\left(j+1,\frac{\hat{\sigma}^2_{h,\max}P\ln n}{\hat{\sigma}^2_{h,i}\lambda P_\delta}\right)\right)$$

$$\to 0.$$

Next, we prove $M_n \leq \nu_U(n)+\epsilon$ holds with probability approaching one. First in [6], we shown that for any $\delta > 0$ and for sufficiently large, m, the best receiver should have the equivalent noise variance $N_1 \leq P_\delta/\alpha^m$. Otherwise, if $\min_{1\leq i\leq n} N_i > P_\delta/\alpha^m$, as shown for Rayleigh fading channels, the total power constraint or the minimum-rate constraint is violated.

Therefore, to show that

$$\mathbb{P}(M_n \leq \nu_U(n)+\epsilon) \to 1,$$

we only need to show that

$$\mathbb{P}(N_1 \leq P_\delta/\alpha^{\nu_U(n)+\epsilon}) \to 0.$$

Let $p_1 = \mathbf{F}(P_\delta/\alpha^{\nu_U(n)+\epsilon})$. Then, $(1-p_1)^n$ is the probability that all the receivers have equivalent noise variance greater than $P_\delta/\alpha^{\nu_U(n)+\epsilon}$. Hence,

$$\mathbb{P}(N_1 \leq P_\delta/\alpha^{\nu_U(n)+\epsilon}) = 1-(1-p_1)^n, \tag{24}$$

which tends to zero if and only if (Eq.21) holds.

As $n\to\infty$, $g(n)\to 0$ and

$$(1-g(n))^{g^{-1}(n)} \to e^{-1}.$$

Hence, (24) tends to zero if (Eq.22) goes to zero. If $c < \frac{\hat{\sigma}^2_{h,\max}P}{\hat{\sigma}^2_{h,i}\lambda P_\delta}$, the expansion of incomplete gamma function can be applied to the first summation of [6, Eq. (23)]. Hence, using $j! \geq (j/2)^{j/2}$ and defining

$$j_0 = 2\left(\frac{\hat{\mu}^2\hat{\sigma}^2_{h,\max}P}{\hat{\sigma}^2_{h,\max}P-\hat{\sigma}^2_{h,i}\lambda P_\delta}\right)^2,$$

the first summation of (22) goes to zero as, n, tends to infinity by choosing $\delta < (\frac{\hat{\sigma}^2_{h,\max}}{\hat{\sigma}^2_{h,i}}\alpha^\epsilon - 1)P$.

Using Stirling's approximation for sufficiently large $c\ln n$, the second summation in (22), is basically similar to [6, Eqs. (25), (26)] tends to zero as, n, goes to infinity. Hence, according to [6, Eqs. (24), (25)], $n\,g(n)\to 0$. ■

The results obtained for Rayleigh and Rician fading channels can also be extended to Nakagami fading channels with a constant shift which is a function of the minimum rate and distribution parameters.

Remark 4.1: Consider independent Nakagami fading channels for different receivers in the broadcast channel. Assume estimated channels for different receivers are independent Nakagami fading channels with channel gains $|\hat{h}_i| \sim Nakagami(\hat{m}_i,\hat{\Omega}_i)$, $i = 1,\ldots,n$ and for any $\epsilon > 0$, the maximum number of active receivers, M_n, is bounded as (11) where

$$\nu_L(n) \triangleq \ln((\frac{\hat{\Omega}_i}{\hat{m}_i})_{\min}\frac{P\ln n}{\sigma^2})/R_{\min} \tag{25}$$

and

$$\nu_U(n) \triangleq \ln((\frac{\hat{\Omega}_i}{\hat{m}_i})_{\max}\frac{P\ln n}{\sigma^2})/R_{\min}. \tag{26}$$

Proof: The proof is similar to the proof of Theorem 4.2 in [6], and Theorem 4.1 in this paper when the Nakagami fading model is considered and the receivers only know estimated channel gains. ■

V. Numerical Results

In the simulation results, it is assumed that the ambient noise variance equals one, the channel gain variance is equal to one for Rayleigh fading and two for Rician fading with the expected value equals one, and the channel bandwidth is 100 Hz. The upper and lower bounds of the channel estimator variance are $\hat{\sigma}^2_{h,\max} = 2\sigma^2_{h,i}$ and $\hat{\sigma}^2_{h,\min} = 0.5\sigma^2_{h,i}$. Fig. 1 shows the number of active receivers (i.e. the user capacity) versus the total number of receivers for equals 25 and 50 Kbps and transmitted power changes linearly with the number of users in the system. In Fig. 1, it can be seen that the user capacity is inversely proportional to the required minimum rate and the number of active users doubles by halving the minimum rate $R_{\min}$. Fig. 1.a and Fig. 1.b are obtained for Rayleigh and Rician fading respectively.

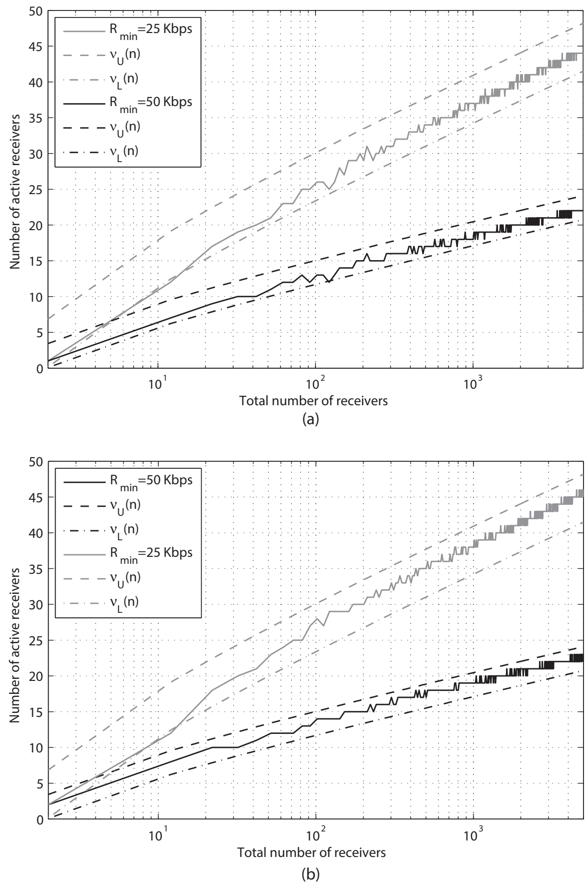

Fig. 1. The optimal number of active receivers versus the total number of users for $R_{\min} = 25, 50$ Kbps and Linearly increasing transmit power (i.e. $P = 10 \log_{10}(n)$ dB). (a) Rayleigh fading and (b) Rician fading.

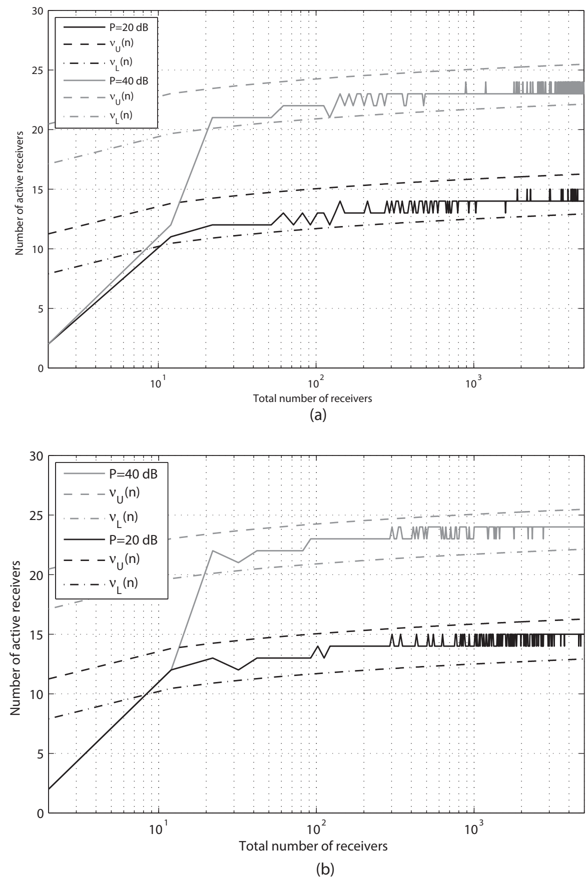

Fig. 2. The optimal number of active receivers versus the total number of users for fixed transmit power $P = 20, 40$ dB. (a) Rayleigh fading and (b) Rician fading.

Fig. 2 indicates the user capacity of broadcast channels for constant transmitted power of 20 and 40 dB. In Fig. 2, it is clear that the user capacity increases logarithmically by increasing transmitted power. Fig. 2.a and Fig. 2.b are obtained for Rayleigh and Rician fading respectively.

Fig. 3 illustrates the number of active receivers histogram for 10000 Monte Carlo runs when the total number of users is fixed. In Fig. 3.a, the total number of users equals 25 and the user capacity equals 12.93 based on Theorem 2.1 in [5] and considering variations of the channel estimator variance, the user capacity changes between 11.54 and 14.32 based on Theorem 4.1. In Fig. 3.b, the total number of users equals 1000 and the user capacity equals 14.46 based on Theorem 2.1 in [5] and considering variations of the channel estimator variance, the user capacity changes between 13.07 and 15.84 based on Theorem 4.1. It can be seen that by increasing the total number of users, the user capacity converges to the theoretical values with high probability.

VI. Conclusion

In [5], [6], it is shown that by considering true CSI for all receivers in the broadcast channel and common fading distributions such as Rayleigh, Rice, and Nakagami, the user capacity scales double logarithmically with the total number of users and the achieving bound is asymptotically tight. In this paper, considering variations of the channel estimator variance for different users in the broadcast channel, there is a gap between the upper and lower bounds of the user capacity; however, it is shown that the user capacity still obeys a double-logarithmic scaling law. If the channel estimator variance remains constant, the upper bound meets the lower bound asymptotically. Simulations indicate that numerical results converge to the theoretical bound as the total number of users increases and show the double-logarithmic scaling law of the user capacity.

VII. Acknowledgments

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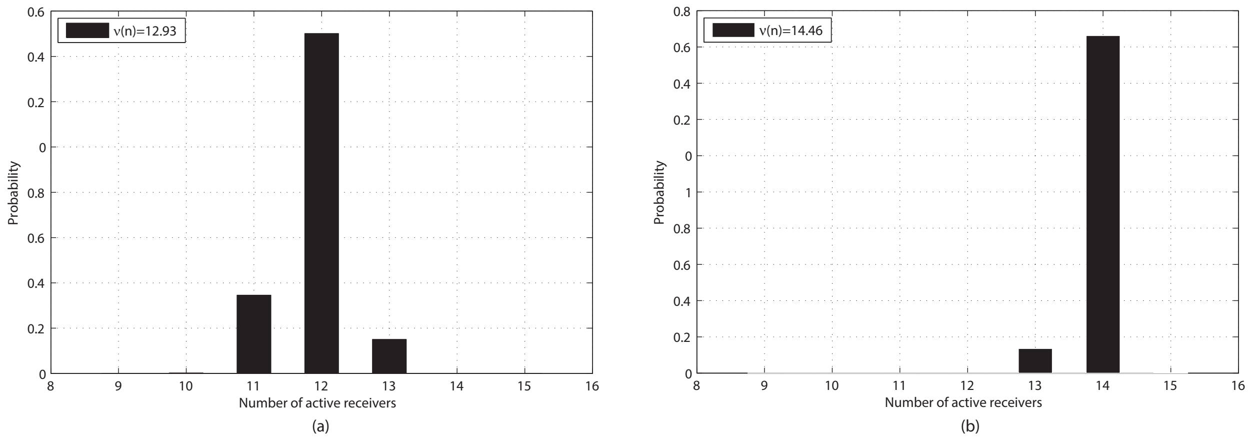

Fig. 3. The histogram of the number of active receivers for $R_{\min} = 50\,\text{Kbps}$, SNR $= 20\,\text{dB}$, and (a) $n = 25$ and (b) $n = 1000$.

Appendix A

Lemma A.1: Consider a Rician random variable $X \sim Rice(\xi, \kappa)$. For any positive r , if $Y = rX$, Y is distributed as $Rice(r\xi, r\kappa)$.

Proof:

$$
\begin{aligned}
f_Y(y) &= \frac{1}{|r|} f_X(y/r) = \frac{1}{r} f_X(y/r) \\
&= \frac{y/r}{r\kappa^2} \exp\left(-\frac{y^2/r^2 + \xi^2}{2\kappa^2}\right) I_0\left(\frac{y/r\xi}{\kappa^2}\right) \\
&= \frac{y}{(r\kappa)^2} \exp\left(-\frac{y^2 + (r\xi)^2}{2(r\kappa)^2}\right) I_0\left(\frac{y(r\xi)}{(r\kappa)^2}\right)
\end{aligned}
$$

■

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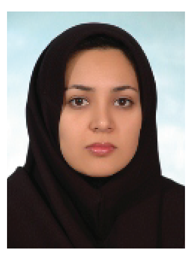

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